2012 KMO Problem 3

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Find all triples (m, p, q) of positive integer m and primes p, q such that $2^m p^2 + 1 = q^5$.

$2^m p^2 = (q-1)(q^4+q^3+q^2+q+1)$, $2 \nmid q$

$q^4+q^3+q^2+q+1 \equiv 1 \pmod 2 \Rightarrow q-1 = 2^m$

i) $v_p(q-1) = 0$ — $q-1 < q^4+q^3+q^2+q+1$

~~ii) $v_p(q-1) = 1$~~ $\Longrightarrow 2^m p < p$ ✗

~~iii) $v_p(q-1) = 2$~~ $\Longrightarrow 2^m p^2 < 1$ ✗

$q-1 = 2^m$, $q^4+q^3+q^2+q+1 = p^2$

$q = 2^m + 1$ < $2 \mid m$ / $2 \nmid m$

Case 1. $2 \nmid m$

$q = (2+1)(2^{m-1} - 2^{m-2} + \dots - 2 + 1)$

$m=1$, $q=3$, $81+27+9+3+1 = 121 = p^2 \Rightarrow p = 11$ — **(1, 11, 3)**

Case 2. $2 \mid m$

$m = 2n$

$q = 4^n + 1$ < $2 \mid n$ / $2 \nmid n \Rightarrow n=1, q=5$, $625+125+25+5+1 = p^2 = 781$ ✗

$1+1+1+1+1 \equiv p^2 \pmod 4$

$\therefore p \equiv 1 \pmod 4$

($2 \mid n$:) Let $n = 2l$, $p = 4k+1$

$q = 16^l + 1$, $p^2 = 16k^2 + 8k + 1$

$q^4+q^3+q^2+q+1 \equiv 1+1+1+1+1 \equiv 5 \pmod 8$

$p^2 \equiv 1 \pmod 8$

$\Leftrightarrow$ No solution here !!!

If $m \neq 1$, then $4 \mid m$

(1, 11, 3) is unique.

□

29×29: 261, 580, 841